Mean-square Stability Control for Networked Systems with Stochastic Time Delay

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***Abstract*—This note is concerned with the problem of delay-distribution-dependent stability and stabilization for networked control systems with stochastic network induced delay and data packet dropout. The stochastic delay and data packet dropout are viewed as stochastic time-varying delay without any constraints on its derivative. Which is assumed to be satisfying a interval Bernoulli distribution. Due to the probability of the delay taking value in different intervals, a new approach is given to model the networked control systems. Based on the Lyapunov stability theory, with the linear matrix inequality approach, a new stabilization criterion is obtained. Then the controller is given to make the closed-loop systems mean-square stable. A numerical example is provided to demonstrate the validity of the proposed design approach.**

***Index Terms*—networked control systems;stochastic delay; packet dropout**

I. INTRODUCTION

It is well known that a feedback control system in which all devices(sensors, controllers and actuators) are interconnected by communication networks is called networked control systems(NCSs).Compared with the traditional point-to-point architecture control systems, the networked control systems have the advantages of high mobility, low cost, easy maintenance and reconfiguration. For these advantages, the networked control systems receive more and more attention and has been a very hot research topic [1-4].

However, the network itself is a dynamic system that exhibits characteristics such as network-induced delay and packet losses. The insertion of the communication network induces different forms of time delay uncertainty between sensors, actuators and controllers. These time delays come from the time sharing of the communication medium as well as the computation time required for physical signal coding and communication processing [5-8]. It is well known in control systems that time delays can degrade a system's performance and even cause system instability. Another significant difference between NCSs and standard digital control is the possibility that data may be lost while in transit through the network because of uncertainty and noise. And the worked-induced delay or packet losses are usually stochastic and described by a two-state Markov chain, which is especially common in wireless communication networks such as Bluetooth and WLAN. Different methods have been developed to deal with the above problems [9-11]. The stochastic optimal control of networked control systems whose network-induced delay is shorter than one sampling period is studied using the time-stamp technique [10]. By viewing the time-varying network-induced delay as parameter uncertainty, the stabilization of networked control systems with delay shorter than one sampling period is studied and sufficient conditions expressed in linear matrix inequality are presented in [11].

In this paper, our objective is to consider the problem of delay-distribution-dependent stability for networked control systems with stochastic delay and data packet dropout. A new approach is given to model the NCSs. Based on the Lyapunov stability theory, a new stabilization criterion is given.

II. PROBLEM FORMULATION

Consider the following control system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(t_0) &= x_0 \end{aligned} \qquad (1)$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, $y(t) \in R^r$ is the controlled output, x_0 is an constant vector. A, B, C, D are known real constant matrices with appropriate dimensions.

Throughout this note, we suppose that all the system's states are available for a state feedback control. In the presence of the control network, which is shown in fig.1, data transfers between the controller and the remote system, e.g., sensors and actuators in a distributed control system will induce network delay in addition to the controller proceeding delay.

In this note we make the following assumptions:
Assumption 1: Sensor and controller are clock-driven.
Assumption 2: Actuator is event-driven.

In Fig.1, τ^{sc} is the communication delay between the sensor and the controller; τ^{c} is the computational delay in the controller; τ^{ca} is the communication delay between the controller and the actuator. Here we suppose that $\tau_k = \tau^{sc} + \tau^{c} + \tau^{ca} \in [0, \bar{\tau}]$ is stochastic network-induced delay, where $\bar{\tau}$ is a constant. $d(k) \in [0, \bar{d}]$ representing the stochastic data dropout number between the sensor and the actuator.

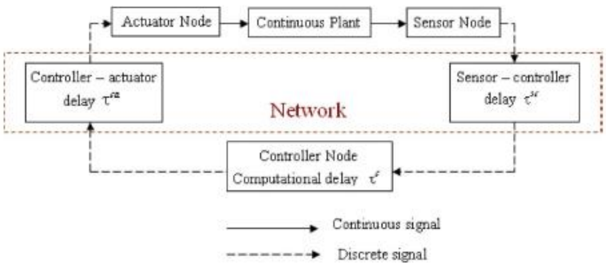

Figure 1. A typical networked control system

We introduce the stochastic delay $h(t)$ to denote the stochastic delay and data packet dropout, where

$$h(t) = t - t_k + d(k)T + \tau_k$$

where $h(t) \in [0, \tau_2]$, $\tau_2 = (1+\bar{d})T + \bar{\tau}$.

With the state feedback controller

$$u(t) = Kx(t) \tag{2}$$

where $K \in R^{m\times n}$ is a constant matrix to be obtained.

We know that

$$u(t) = K\bar{x}(t_k) = Kx(t_k - d(k)T - \tau_k) = Kx(t - h(t))$$

Inserting the above controller into system(1), we obtain the closed system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + BKx(t - h(t)) \\ &y(t) = Cx(t) + Du(t) \quad t \in [t_k, t_{k+1}] \quad k = 1, 2, \cdots \\ &x(t) = \phi(t) \qquad t \in [t_0 - \tau_2,\ t_0] \end{aligned} \tag{3}$$

where $h(t)$ is stochastic on $[0,\ \tau_2]$ without any constraint on its derivative. The initial condition of the state $x(t)$ on $t \in [t_0 - \tau_2,\ t_0]$ is supplemented as $x(t) = \phi(t)$, $t \in [t_0 - \tau_2,\ t_0]$, $\phi(t_0) = x_0$, where $\phi(t)$ is a continuous function on $[t_0 - \tau_2,\ t_0]$.

It is assumed that there exists a constant $\tau_1 \in [0,\ \tau_2)$ such that the probability of $h(t)$ taking values in $[0,\ \tau_1)$ and $[\tau_1,\ \tau_2]$ can be observed. In order to employ the information of the probability distribution of the delay in the system model, the following sets are proposed firstly

$$\Omega_1 = \{t : h(t) \in [0,\ \tau_1)\} \quad \Omega_2 = \{t : h(t) \in [\tau_1,\ \tau_2]\}$$

Obviously, $\Omega_1 \bigcup \Omega_2 = R^+$ and $\Omega_1 \bigcap \Omega_2 = \Phi$.

Then we define two functions as:

$$h_1(t) = \begin{cases} h(t) & t \in \Omega_1 \\ 0 & t \notin \Omega_1 \end{cases} \quad h_2(t) = \begin{cases} h(t) & t \in \Omega_2 \\ \tau_1 & t \notin \Omega_2 \end{cases}$$

Corresponding to $h(t)$ taking values in different intervals, a stochastic variable $\beta(t)$ is defined

$$\beta(t) = \begin{cases} 1 & t \in \Omega_1 \\ 0 & t \in \Omega_2 \end{cases}$$

Where we suppose that $\beta(t)$ is a Bernoulli distributed sequence with $\Pr ob\{\beta(t) = 1\} = E\{\beta(t)\} = \beta$, where $\beta \in [0,1]$ is a constant.

By using the new functions $h_1(t), h_2(t)$ and stochastic variable $\beta(t)$, the systems(3) can be equivalently written as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + \beta(t)BKx(t - h_1(t)) + (1 - \beta(t))BKx(t - h_2(t)) \\ &y(t) = Cx(t) + Du(t) \quad t \in [t_k, t_{k+1}] \quad k = 1, 2, \cdots \\ &x(t) = \phi(t) \qquad t \in [t_0 - \tau_2,\ t_0] \end{aligned} \tag{4}$$

III. MAIN RESULTS

Lemma 1 [2] For any vectors a, b and matrices N, X, Y, Z with appropriate dimensions, if the following matrix inequality holds

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0$$

then we have

$$-2a^T Nb \leq \inf_{X,Y,Z} \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - N \\ Y^T - N^T & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Lemma 2 [4] The LMI $\begin{bmatrix} Y(x) & W(x) \\ * & R(x) \end{bmatrix} > 0$ is equivalent to

$$R(x) > 0, Y(x) - W(x)R^{-1}(x)W^T(x) > 0$$

where $Y(x) = Y^T(x), R(x) = R^T(x)$ depend on x.

Theorem 1. For the networked control systems (4), if there exist positive-definite matrices $P, Q_1, Q_2 \in R^{n\times n}$, matrices $K \in R^{m\times n}$, X_i, Y_i $i = 1, 2$ with appropriate dimensions , such that the following matrix inequalities hold

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & \Theta_{33} \end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} X_1 & Y_1 \\ * & Q_1 \end{bmatrix} \geq 0 \tag{6}$$

$$\begin{bmatrix} X_2 & Y_2 \\ * & Q_2 \end{bmatrix} \geq 0 \tag{7}$$

where

$$\begin{aligned}
\Theta_{11} &= PA + A^T P + \tau_1 A^T Q_1 A + \tau_1 X_{111} + \tau_2 A^T Q_2 A \\
&\quad + \tau_2 X_{211} + Y_{11} + Y_{11}^T + Y_{21} + Y_{21}^T \\
\Theta_{12} &= P\beta BK + \tau_1 A^T Q_1 \beta BK + \tau_1 X_{112} + \tau_2 \beta A^T Q_2 BK \\
&\quad + \tau_2 X_{212} - Y_{11} + Y_{12}^T + Y_{22}^T \\
\Theta_{13} &= (1-\beta)PBK + \tau_1 (1-\beta) A^T Q_1 BK + \tau_1 X_{113} \\
&\quad + \tau_2 (1-\beta) A^T Q_2 BK + \tau_2 X_{213} - Y_{21} + Y_{13}^T + Y_{23}^T \\
\Theta_{22} &= \tau_1 K^T B^T Q_1 \beta BK + \tau_1 X_{122} + \tau_2 \beta K^T B^T Q_2 BK \\
&\quad + \tau_2 X_{222} - Y_{12} - Y_{12}^T \\
\Theta_{23} &= \tau_1 X_{123} + \tau_2 X_{223} \\
\Theta_{33} &= \tau_1 (1-\beta) K^T B^T Q_1 BK + \tau_1 X_{133} \\
&\quad + \tau_2 (1-\beta) K^T B^T Q_2 BK + \tau_2 X_{233} - Y_{23} - Y_{23}^T
\end{aligned}$$

with the controller (2), the network control systems(4) is mean-square stable.

Proof Choose a Lyapunov functional candidate for the system(4) as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{8}$$

where

$$V_1(t) = x^T(t) P x(t)$$

$$V_2(t) = \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_1 \dot{x}(s) ds d\theta$$

$$V_3(t) = \int_{-\tau_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds d\theta$$

where P, Q_1, Q_2 positive-definite matrices with appropriate dimensions.

Then, along the solution of system (4) we have

$$\dot{V}_1(t) = 2x^T(t) P \bar{A} \xi(t) \tag{9}$$

where

$$\xi(t) = [x^T(t) \quad x^T(t-h_1(t)) \quad x^T(t-h_2(t))]$$

$$\bar{A} = [A \quad \beta(t)BK \quad (1-\beta(t))BK]$$

$$\dot{V}_2(t) = \tau_1 \xi^T(t) \bar{A}^T Q_1 \bar{A} \xi(t) - \int_{t-\tau_1}^{t} \dot{x}^T(s) Q_1 \dot{x}(s) ds \tag{10}$$

and

$$\dot{V}_3(t) = \tau_2 \xi^T(t) \bar{A}^T Q_2 \bar{A} \xi(t) - \int_{t-\tau_2}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds \tag{11}$$

We know

$$x(t) - x(t-h_1(t)) - \int_{t-h_1(t)}^{t} \dot{x}(s) ds = 0$$

$$x(t) - x(t-h_2(t)) - \int_{t-h_2(t)}^{t} \dot{x}(s) ds = 0$$

With two $4n \times n$ matrices

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

By the lemma 1, we obtain

$$\begin{aligned}
0 &\le 2\xi^T(t) N[x(t) - x(t-h_1(t))] \\
&\quad + \int_{t-h_1(t)}^{t} \begin{bmatrix} \xi(t) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} X_1 & Y_1 - N \\ * & Q_1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{x}(s) \end{bmatrix} ds \\
&\le 2\xi^T(t) Y_1 [x(t) - x(t-h_1(t))] + \tau_1 \xi^T(t) X_1 \xi(t) \\
&\quad + \int_{t-\tau_1}^{t} \dot{x}^T(s) Q_1 \dot{x}(s) ds
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
0 &\le 2\xi^T(t) Y_2 [x(t) - x(t-h_2(t))] + \tau_2 \xi^T(t) X_2 \xi(t) \\
&\quad + \int_{t-\tau_2}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds
\end{aligned} \tag{13}$$

where

$$X_i = \begin{bmatrix} X_{i11} & X_{i12} & X_{i13} \\ * & X_{i22} & X_{i23} \\ * & * & X_{i33} \end{bmatrix}, Y_i = \begin{bmatrix} Y_{i1} \\ Y_{i2} \\ Y_{i3} \end{bmatrix} \; i = 1, 2$$

With the (9-13),we can obtain

$$E\{\dot{V}(t)\} = \xi^T(t) \Theta \xi(t)$$

With the Lyapunov stability theorem and the inequality (5), we know that the system (4) is mean-square stable.

Theorem 2 For the networked control systems (4), if there exist positive-definite matrices $\bar{P}, \bar{Q}_1, \bar{Q}_2 \in R^{n \times n}$, matrices $\bar{K} \in R^{m \times n}$, $\bar{X}_i, \bar{Y}_i, i = 1, 2$ with appropriate dimensions , such that the following linear matrix inequalities hold

$$\begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \tau_1 \beta \bar{P} A^T & \tau_1 (1-\beta) \bar{P} A^T & \tau_2 \beta \bar{P} A^T & \tau_2 (1-\beta) \bar{P} A^T \\
* & \Xi_{22} & \Xi_{23} & \tau_1 \beta \bar{K}^T B^T & 0 & \tau_2 \beta \bar{K}^T B^T & 0 \\
* & * & \Xi_{33} & 0 & \tau_1 (1-\beta) \bar{K}^T B^T & 0 & \tau_2 (1-\beta) \bar{K}^T B^T \\
* & * & * & -\tau_1 \beta \bar{Q}_1 & 0 & 0 & 0 \\
* & * & * & * & -\tau_1 (1-\beta) \bar{Q}_1 & 0 & 0 \\
* & * & * & * & * & -\tau_2 \beta \bar{Q}_2 & 0 \\
* & * & * & * & * & * & -\tau_2 (1-\beta) \bar{Q}_2
\end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} X_1 & Y_1 \\ * & Q_1 \end{bmatrix} \ge 0 \tag{15}$$

$$\begin{bmatrix} X_2 & Y_2 \\ * & Q_2 \end{bmatrix} \ge 0 \tag{16}$$

where

$$\Xi_{11} = -A\bar{P} + \bar{P}A^T + \tau_1 A^T Q_1 A + \tau_1 \bar{X}_{111} + \tau_2 \bar{X}_{211} + \bar{Y}_{11} + \bar{Y}_{11}^T + \bar{Y}_{21} + \bar{Y}_{21}^T$$
$$\Xi_{12} = \beta B\bar{K} + \tau_1 \bar{X}_{112} + \tau_2 \bar{X}_{212} - \bar{Y}_{11} + \bar{Y}_{12}^T + \bar{Y}_{22}^T$$
$$\Xi_{13} = (1-\beta)B\bar{K} + \tau_1 \bar{X}_{113} + \tau_2 \bar{X}_{213} - \bar{Y}_{21} + \bar{Y}_{13}^T + \bar{Y}_{23}^T$$
$$\Xi_{22} = \tau_1 \bar{X}_{122} + \tau_2 \bar{X}_{222} - \bar{Y}_{12} - \bar{Y}_{12}^T$$
$$\Xi_{23} = \tau_1 \bar{X}_{123} + \tau_2 \bar{X}_{223}$$
$$\Xi_{33} = \tau_1 \bar{X}_{133} + \tau_2 \bar{X}_{233} - \bar{Y}_{23} - \bar{Y}_{23}^T$$

with the controller $u(t) = \bar{K}\bar{P}^{-1}x(t)$, the systems (4) is mean-square stable.

Proof Now we proof that the inequality (5) is equivalent to the inequality (14). Obviously, by the lemma2, (5) is equivalent to

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \tau_1\beta A^T Q_1 & \tau_1(1-\beta)A^T Q_1 & \tau_2\beta A^T Q_2 & \tau_2(1-\beta)A^T Q_2 \\ * & \Omega_{22} & \Omega_{23} & \tau_1\beta K^T B^T Q_1 & 0 & \tau_2\beta K^T B^T Q_2 & 0 \\ * & * & \Omega_{33} & 0 & \tau_1(1-\beta)K^T B^T Q_1 & 0 & \tau_2(1-\beta)K^T B^T Q_2 \\ * & * & * & -\tau_1\beta Q_1 & 0 & 0 & 0 \\ * & * & * & * & -\tau_1(1-\beta)Q_1 & 0 & 0 \\ * & * & * & * & * & -\tau_2\beta Q_2 & 0 \\ * & * & * & * & * & * & -\tau_2(1-\beta)Q_2 \end{bmatrix} < 0 \quad (17)$$

Where

$$\Omega_{11} = PA + A^T P + \tau_1 X_{111} + \tau_2 X_{211} + Y_{11} + Y_{11}^T + Y_{21} + Y_{21}^T$$
$$\Omega_{12} = \beta PBK + \tau_1 X_{112} + \tau_2 X_{212} - Y_{11} + Y_{12}^T + Y_{22}^T$$
$$\Omega_{13} = (1-\beta)PBK + \tau_1 X_{113} + \tau_2 X_{213} - Y_{21} + Y_{13}^T + Y_{23}^T$$
$$\Omega_{22} = \tau_1 X_{122} + \tau_2 X_{222} - Y_{12} - Y_{12}^T$$
$$\Omega_{23} = \tau_1 X_{123} + \tau_2 X_{223}$$
$$\Omega_{33} = \tau_1 X_{133} + \tau_2 X_{233} - Y_{23} - Y_{23}^T$$

Pre- and Post-multiplying the inequality(17) by

$$diag\{P^{-1} \quad P^{-1} \quad P^{-1} \quad I \quad Q_1^{-1} \quad Q_1^{-1} \quad Q_2^{-1} \quad Q_2^{-1}\}$$

and giving some transformations

$$\bar{P} = P^{-1} \quad \bar{Q}_1 = Q_1^{-1} \quad \bar{Q}_2 = Q_2^{-1} \quad \bar{K} = KP^{-1}$$
$$\bar{X}_{ijk} = P^{-1} X_{ijk} P^{-1} \quad \bar{Y}_{ij} = P^{-1} Y_{ij} P^{-1} \quad i = 1,2; j = 1,2,3$$

We know the inequality (17) is equivalent to (14).

IV. SIMULATIONS

Consider the networked control systems in the form of (4), where

$$A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \beta = 0.4$$

Solving the linear matrix inequality (14-16), we can obtain the gain matrix K of the stabilizing controller $u(t)$

$$K = [-0.5692 \quad 1.5781]$$

From the theorem 2, we know that the systems (4) is mean-square stable.

ACKNOWLEDGMENT

The author would like to thank the associate editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality and the presentation of the paper.

This work was supported by National Nature Science Foundation under Project (61073065), Henan Province Science and Technology Key Project (172102210162), The National College Students' Innovative Entrepreneurial Training Project (201610479045), Anyang Normal University Innovation Foundation Project.

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